PIMCO

Sovereign Assets, Optimal Growth and Volatility Pumping

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Executive Summary

- In this paper, we review the idea of volatility pumping and discuss how it helps boost assets of sovereign wealth funds.
- Asset diversification leads to lower portfolio volatility which, in turn, increases expected growth of sovereign portfolios.
- The benefits of volatility pumping increase with the original portfolio volatility.

1 INTRODUCTION

Can sovereign funds structure their assets – financial or real – with a view to boosting these assets' growth? We will address this question after discussing the basics of volatility pumping.

Volatility pumping is an asset allocation and trading strategy that makes use of the following fact: Even though the correct way to measure the performance of a financial asset is its geometric mean return, its expectation at any point in time is its arithmetic mean return, which is always equal to or higher than its geometric mean.

1.1 COMPOUNDED RETURN: A REMINDER

Because the value of a portfolio P_0, continuously compounded at a rate r, yields $P_T = P_0 e^{rT}$ after T periods, it follows that a compounded return r is

$$r = \frac{\log \frac{P_T}{P_0}}{T}. \qquad (1)$$

Specifically, if a dollar is continuously compounded over a year to yield P_T, then the compounded return on that dollar is $\log(P_T)$. In an uncertain world, the expected compounded return, also called compounded growth, is $E\left(\log(P_T)\right)$.

2 VOLATILITY PUMPING

Say we have two uncorrelated assets with zero expected growth. Is it possible to combine them so that the portfolio's expected growth exceeds zero?

2.1 TWO SIMPLE EXAMPLES

Example 1: A risk asset and a riskless asset, both zero-growth. Let us try our hand at building a portfolio with a risk-free asset yielding zero interest and a risk asset that over a year sees its value multiplied by three in the upstate with a probability of 50% or divided by three in the downstate with a probability of 50%. The expected growth rate for the risk asset is

$$\frac{1}{2}\log(3) + \frac{1}{2}\log\left(\frac{1}{3}\right) = 0. \quad (2)$$

Both assets have an expected growth rate of zero. But what happens if we allocate, say, one quarter of a dollar to the risk asset and three quarters of a dollar to the riskless asset? The dollar will be worth either (1/4)*3+(3/4)*1 = 3/2 in the upstate with a probability of 50% or (1/4)*(1/3)+(3/4)*1 = 5/6 in the downstate with a probability of 50%.

The expected growth rate of the portfolio is therefore

$$\frac{1}{2}\log\left(\frac{3}{2}\right) + \frac{1}{2}\log\left(\frac{5}{6}\right) = 0.1157. \quad (3)$$

It appears that diversification generates portfolio growth of 11.57% even though the expected growth of the portfolio's constituents is zero. But we can do better than that. Let us try a 50-50 allocation instead of 25-75; the dollar will be worth (1/2)*3+(1/2)*1 = 2 in the upstate and (1/2)*(1/3)+(1/2)*1 = 2/3 in the downstate.

Now the portfolio's expected growth rate is

$$\frac{1}{2}\log(2)+\frac{1}{2}\log\left(\frac{2}{3}\right)=0.1438. \quad (4)$$

Equal allocation to the risk assets and riskless assets further improves portfolio growth.

As shown in the preceding example, diversification is an engine of financial growth. How so? Diversification reduces volatility, which in turn boosts growth. It is well known in finance that lower volatility translates into higher growth. In Appendix 1, we show that the expected growth rate, or compounded return, of an asset increases as the asset volatility decreases. Specifically:

$$Growth\ rate\ of\ portfolio = Arithmetic\ return - \frac{1}{2}Variance\ of\ return. \quad (5)$$

In the next example, we discuss a similar instance of volatility pumping in a portfolio of two risk assets.

Example 2: Two uncorrelated risk assets, both zero-growth. Consider now two uncorrelated risk assets that see their value either tripled or divided by three with equal probability. As noted in Example 1, the expected growth rate of these assets, taken individually, is zero. But if we constitute an equal-weight portfolio, we have four states of the world for both assets: (upstate, upstate), (upstate, downstate), (downstate, upstate) and (downstate, downstate). The expected growth rate of this portfolio is

$$\frac{1}{4}\log\left(\frac{3}{2}+\frac{3}{2}\right)+\frac{1}{4}\log\left(\frac{3}{2}+\frac{1}{6}\right)+\frac{1}{4}\log\left(\frac{1}{6}+\frac{3}{2}\right)+\frac{1}{4}\log\left(\frac{1}{6}+\frac{1}{6}\right)=0.2554. \quad (6)$$

The portfolio is expected to grow by 25.54% over one year, even though each of its two constituents will have zero growth.

2.2 OPTIMAL PORTFOLIO GROWTH WITH A RISK ASSET AND A RISKLESS ASSET

In Example 1, the expected growth of the portfolio increased as the allocation to the risk asset rose from 25% to 50%. A related question is whether there is an optimal portfolio that maximizes the expected growth. The answer is yes. If the problem is to allocate a portfolio to a risk asset and a zero-yielding riskless asset, and if the risk asset value is multiplied by m with probability p and by $1/m$ with probability $(1\text{-}p)$, then the optimal allocation to the risk asset is

$$\alpha^* = \frac{p(m+1)-1}{m-1}. \quad (7)$$

The proof is in Appendix 2. Note that for p = 1/2 the risk asset has an expected growth of zero and the risk asset allocation is α^*= 1/2. In this case, optimal growth is

$$g^* = \frac{1}{2}\log\left(\frac{1}{2}+\frac{m}{2}\right)+\frac{1}{2}\log\left(\frac{1}{2}+\frac{1}{2m}\right)=\frac{1}{2}\log\left(\frac{1}{2}+\frac{m}{4}+\frac{1}{4m}\right). \quad (8)$$

Recall that m represents the magnitude of the asset price move. Effectively, it is a measure of volatility.

Exhibit 1 shows the optimal growth measured in basis points (bps) as a function of the parameter m.

As this exhibit indicates, the growth numbers induced by volatility pumping are only meaningful for highly volatile assets.

Exhibit 1: Optimal portfolio growth as a function of volatility

Portfolio with a risk asset and a riskless asset, both with zero average compounded return.

m	Optimal growth (bps)
1.00	0
1.05	3
1.10	11
1.15	24
1.20	41
1.25	62
1.30	86
1.35	112
1.40	141
1.45	172
1.50	204
2.00	589
3.00	1,438
4.00	2,231
5.00	2,939

Source: PIMCO

2.3 OPTIMAL PORTFOLIOS AND VOLATILITY PUMPING WITH MULTIPLE ASSETS

We can extend this analysis to multiple assets and a continuous time framework. Two results are worthy of note. First, if we are restricted to a risk asset and a riskless asset, then, in an optimal portfolio,

$$Risk\ asset\ share = \frac{Sharpe\ ratio\ of\ risk\ asset}{Volatility\ of\ risk\ asset\ returns} \tag{9}$$

Appendix 2 proves this result and extends it to multiple assets.

Second, as the portfolio is more diversified, the portfolio risk decreases and its expected growth increases. To help clarify the gain from volatility pumping, compare the growth of a single-asset portfolio with that of a multi-asset portfolio. We assume all assets are uncorrelated, with the same expected arithmetic return μ and the same volatility σ. As discussed above, the growth of a single-asset portfolio is $\mu\text{-}(1/2)\sigma^2$. We show in Appendix 2 that the growth of the multi-asset portfolio is $\mu\text{-}(1/2)\sigma^2$. So the gain from volatility pumping is

$$\mu - \frac{\sigma^2}{2n} - \left(\mu - \frac{\sigma^2}{2}\right) = \frac{\sigma^2}{2}\ \frac{n-1}{n}. \tag{10}$$

This is close to $\sigma^2/2$ for large n, which corresponds to the number of risk assets in our portfolio.

2.4 IMPLICATIONS FOR SOVEREIGN ASSET MANAGEMENT

Volatility pumping has important implications for underdiversified commodity-producing countries. For example, consider a single-commodity country that produces oil. Let us analyze the implications for this country's portfolio growth if it swaps its reserves against a portfolio of four equally weighted and correlated commodities: oil, copper, gold and aluminum.

From 10 years of historical daily spot prices, we can construct the appropriate covariance and correlation matrices for the sovereign's returns. We use Brent prices for oil. The volatilities (standard deviation of returns) are given by: Brent oil σ_o=26%, copper σ_c=14.8%, gold σ_g=7.6% and aluminum σ_a=14.9%. The correlations among the commodities are given by: ϱ_{oc}=0.35, ϱ_{og}=0.17 , ϱ_{oa}=0.3, ϱ_{cg}=0.277, ϱ_{ca}=0.634 and ϱ_{ga}=0.214. Thus, the covariance matrix is given by

$$\begin{pmatrix} 0.0680 & 0.0135 & 0.0034 & 0.0177 \\ 0.0135 & 0.0219 & 0.0031 & 0.0140 \\ 0.0034 & 0.0032 & 0.0058 & 0.0024 \\ 0.0117 & 0.0140 & 0.0024 & 0.0222 \end{pmatrix}. \tag{11}$$

The expected growth of the single-commodity portfolio P_t is given by (see Appendix 2 for the derivation):

$$E(g_P) = \frac{dlnP_t}{dt} = \mu - \sigma^2/2. \tag{12}$$

The expected growth of a portfolio of n assets Π_t is given by (see Appendix 2 for the derivation):

$$E(g_\Pi) = \frac{dln\Pi_t}{dt} = r + \alpha^T(\mu\text{-}r\ ) - (1/2)\alpha^T\Sigma\alpha. \tag{13}$$

In our case, the weights of the four commodities are equal to one another, so

$$\alpha^T = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right). \tag{14}$$

The pickup in growth due to volatility pumping can be obtained by subtracting $E(g_P)$ from $E(g_\Pi)$ and inserting the values for σ_o^2 for σ^2 and the values for Σ from the covariance matrix above:

$$(\sigma_o^2)/2\text{-}(1/2)\alpha^T\Sigma\alpha = 2.735\%. \tag{15}$$

The pickup is clearly substantial.

How would we engineer this swap in practice? Obviously, there would be a number of issues involved, including the depth and liquidity of commodity markets, the feasibility/legality of macro swaps at the sovereign level and the search for a substitute portfolio. Avenues for resolving these problems would include commodity swaps or supranational clearing banks. Given the magnitude of the benefit, it seems to us to be a worthwhile exercise.

3 CONCLUSION

In this paper, we explored the potential benefits of volatility pumping for sovereign wealth funds. This asset allocation strategy makes use of the fact that while the expectation of a financial asset's performance is its arithmetic mean return, the correct measure of its performance is its geometric mean return. Because it is a mathematical fact that the arithmetic mean return is greater than or equal to the geometric mean return, we may benefit from the difference to increase the growth of the sovereign portfolio.

The effect of pumping is most significant when the variance of the original portfolio is high; as we have shown, the gain from pumping is a positive function of volatility. As the number of assets in the portfolio increases, the gain from pumping approaches 50% of the variance of risk asset returns. Hence, the higher the variance, the larger the benefit.

Using the simple example of a single-commodity-producing sovereign, we analyzed the implications for the country's portfolio growth of "swapping" its reserves against a portfolio of four equally weighted and correlated commodities: oil, copper, gold and aluminum. Given the substantial growth pickup, we believe the strategy could be highly attractive for sovereign wealth funds to use in their asset-liability management activities. The swap portfolio does not have to be limited to other commodities, as in our example. It can include equity indices, real or nominal bond indices, real estate or any alternative asset class with variance and correlation characteristics that will maximize the growth pickup due to volatility pumping.

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Appendix 1: Geometric and arithmetic growth rates

Let r be the arithmetic rate of return. Then $\log(1+r)$ is the geometric rate of return – what we call the growth rate. A second-order Taylor expansion of $\log(1+r)$ around the expectation of r – which we call μ – will give

$$\log(1+r) \approx \log(1+\mu) + \frac{r-\mu}{1+r} - \frac{(r-\mu)^2}{2\,(1+\mu)^2}. \tag{A1.1}$$

Taking expectations on both sides, we obtain

$$E\,(g) = E\,(\log(1+r)) \approx \log\,(1+\mu) - \frac{V(r)}{2(1+\mu)^2}. \tag{A1.2}$$

Therefore, as mentioned in the main text, a lower volatility of asset returns implies a higher expected growth.

When prices follow a geometric Brownian motion,

$$\frac{dP_t}{P_t} = \mu dt + \sigma dW_t. \tag{A1.3}$$

Then, by Ito's lemma,

$$d\log P_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t. \tag{A1.4}$$

Hence, expected growth is equal to

$$E(g) = \frac{d\log P_t}{dt} = \mu - \frac{\sigma^2}{2}; \tag{A1.5}$$

meaning that expected growth (or expected geometric return) is equal to the drift of the return (or expected arithmetic return) minus the variance of returns.

Appendix 2: Optimal growth portfolios with a risk asset and a riskless asset in discrete time

We want to allocate a portfolio between a risk asset and a zero-yielding riskless asset. The risk asset value is multiplied by m (m is greater than 1) with probability p and by $1/m$ with probability $(1-p)$. Call α the share of the risk asset in the portfolio. In the upstate, the portfolio delivers $\alpha m+(1-\alpha)$; in the downstate, $\alpha/m+(1-\alpha)$. To maximize growth, we want to choose α to optimize:

$$E(g) = p\log\left(\alpha m+(1-\alpha)\right) + (1-p)\log\left(\alpha/m+(1-\alpha)\right). \quad \text{(A2.1)}$$

The first-order condition is

$$\frac{p(m-1)}{\alpha^*(m-1)+1} = \frac{(1-p)(1-1/m)}{1+\alpha^*(1/(m-1)-1)}. \quad \text{(A2.2)}$$

The optimal allocation to the risk asset is hence

$$\alpha^* = (p(m+1)-1)/(m-1). \quad \text{(A2.3)}$$

First, we build a portfolio with a risk asset and a riskless asset. The risk asset price S follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad \text{(A2.4)}$$

whereas the riskless asset price B follows the simple dynamic

$$\frac{dB_t}{B_t} = rdt. \quad \text{(A2.5)}$$

If the allocation to the risk asset is α, then the portfolio dynamic is

$$\frac{d\pi_t}{\pi_t} = (r+\alpha(\mu-r))dt+\alpha\sigma dW_t \quad \text{(A2.6)}$$

where π_t is the value of the portfolio. To maximize the growth of this portfolio, we first obtain an expression of $\log\pi_t$ using Ito's lemma:

$$d\log\pi_t = \left(r+\alpha(\mu-r)-\left(\alpha^2\sigma^2/2\right)\right)dt+\alpha\sigma dW_t. \quad \text{(A2.7)}$$

We now pick α to maximize the dt term:

$$\alpha^* = (\mu-r)/\sigma^2. \quad \text{(A2.8)}$$

The optimal risk asset allocation is the asset risk premium divided by the variance of the return. It is also the Sharpe ratio divided by the volatility. The reader can check that the optimal excess growth over the risk-free rate is half the square of the risk asset Sharpe ratio by replacing α^* in the growth equation. Again, by replacing the expression of α^* in the portfolio dynamics equation, one can see that the volatility of the portfolio is the risk asset Sharpe ratio.

Looking at multiple assets following a multivariate geometric Brownian motion with parameters (μ,Σ), the problem, as before, is to find optimal weights by maximizing

$$r+\alpha^{\mathsf{T}}(\mu-\mathbf{r})-\frac{1}{2}\alpha^{\mathsf{T}}\Sigma\alpha \quad \text{(A2.9)}$$

where α is an $(n\times 1)$ vector of optimal weights in n risk assets, μ is an $(n\times 1)$ vector of risk asset returns, Σ is the $(n\times n)$ covariance matrix (that is, the variances and covariances of risk returns) and $\mathbf{r}$ is an $(n\times 1)$ constant vector of identical values r (risk-free rates). Setting the differential of the above expression with respect to α equal to zero, we find the optimal allocation to risk assets is

$$\alpha^* = \Sigma^{-1}(\mu-\mathbf{r}). \quad \text{(A2.10)}$$

Last, when asset returns are independent and, across all risk assets, Sharpe ratios are identical and volatilities are equal to σ, all risk asset weights are $1/n$, so the expression of the variance is trivially equal to

$$\frac{1}{2}\alpha^{\mathsf{T}}\Sigma\alpha = \frac{\sigma^2}{2n}. \quad \text{(A2.11)}$$

This paper contains hypothetical analysis. Results shown may not be attained and should not be construed as the only possibilities that exist. The analysis reflected in this information is based upon a set of assumptions believed to be reasonable at time of creation. Actual returns will vary. Forecasts, estimates, and certain information contained herein are based upon proprietary research and should not be considered as investment advice or a recommendation of any particular security, strategy or investment product.

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